Research Program

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A subgroup $A \leq B$ of an abelian group B is said to be a *pure* subgroup if, for every natural number n, the equality

$$A \cap nB = nA$$

holds: every element of A divisible by n in B, is also divisible by n in A. It means that the inclusion of A in B is a strong form of inclusion, one that respects local context. Such strengthened forms of structural inclusion pervade mathematics, and this definition for abelian groups, due to Prüfer [59], was generalized by P.M. Cohn [13] to the setting of modules over an associative ring R, by the decree that a morphism $f : {}_RM \to {}_RN$ be a *pure* monomorphism if for every right R-module X_R, the induced morphism

$$1_X \otimes f : X \otimes_R M \to X \otimes_R N$$

of abelian groups is a monomorphism. Because the tensor product is a *continuous* functor, that is, commutes with direct limits, and every module is a direct limit of finitely presented modules [49], it suffices to take X_R finitely presented in the definition of purity.

A left R-module M is called *pure injective,* or *algebraically compact,* if every pure monomorphism $f : M \to N$ has a retraction. The existence of pure injective envelopes in the category R-Mod of all (left) R-modules has been proved using various techniques [47, 64, 66]. Every *injective* module is pure injective. On the other hand, if k is a field, and R a k-algebra, then every finite dimensional representation of R is also pure injective. The indecomposable algebraically compact abelian groups ($R = \mathbb{Z}$) are [18, 45]: the finite indecomposable groups $\mathbb{Z}(p^n)$, p a prime and $n > 0$; the indecomposable divisible (injective) groups $\mathbb{Z}(p^\infty)$, and $\mathbb{Q}$; and the p-adic completions $\overline{\mathbb{Z}}_p$.

If we denote by mod-R the category of finitely presented right R-modules, the definition of purity suggests that, given a left R-module M, we consider the additive functor

$$- \otimes_R M : \text{mod-R} \to \text{Ab},$$

where Ab denotes the category of abelian groups. The additive functors $F : \text{mod-R} \to \text{Ab}$ themselves form a category, with morphism the natural transformations. This category is denoted by $(\text{mod-R}, \text{Ab})$ and has the property of being a *Grothendieck* category [19, 61]. The notion of a Grothendieck category was introduced to provide a general setting in which one may formulate the notion of *essential* extension, and prove the existence of *injective envelopes.* Thus the entire classical theory developed for injective modules over a ring may be implemented in an entirely new situation. To our fortune, Gruson and Jensen [23] characterized the injective objects of the category $(\text{mod-R}, \text{Ab})$ as the objects of the form $- \otimes_R M$, where M is a pure injective module.

Model theory [55] is the branch of mathematical logic that provides a completely general framework in which to study local context. The notion of an *elementary extension* offers a notion of inclusion that respects local context in any class of structures that may be identified as satisfying some collection of first-order axioms. Given a ring R, the appropriate *language* $\mathcal{L}(R) = (+, -, 0, r)_{r \in R}$ for the study of left R-modules is an expansion of the language of abelian groups by a collection of unary function symbols r, corresponding to the elements of the ring. It is the formal language [56, 66] necessary to express a linear equation with scalars from the ring R acting on the left; as well as the axioms for a left R-module. If M and N are *elementarily equivalent* left R-modules, Sabbagh [60] proved that a monomorphism $f : M \to N$ is pure if and only if it is an elementary extension.

My research involves the application of category-theoretic [3, 4] and model-theoretic [56, 66] techniques to the representation theory of an associative ring R, with emphasis on pure injective, that is, algebraically compact, representations. The research program may be broadly divided into the following three often overlapping pursuits:

(1) the general theory of pure injective modules;
(2) construction and description of pure injective representations of specific rings R; and
(3) pure injective objects in other categorical settings.

There are four general directions in which these pursuits lead my present interests:

A: consequences of the Flat Cover Conjecture;
B: triangulated categories and phantom morphisms;
C: the K-theory of the free abelian category; and
D: pseudo-finite dimensional representations of Lie algebras, Lie groups and quantum groups.

My work on phantom morphisms applies to the general theory of pure injectives (1), although the results are inspired by phenomena that arise in triangulated categories (3). The K-theory of the free abelian category Ab(R) over the ring R also pertains to the general theory of pure injectives (1), because, as is described below, the Serre subcategories of Ab(R) are in bijective correspondence with the open subsets of the Ziegler spectrum of R, a space whose points are the pure injective indecomposable R-modules. Part of our work on the consequences of the Flat Cover Conjecture includes the study of pure injective objects in additive accessible categories (3).

Consequences of the Flat Cover Conjecture. In 2001, Bican, El Bashir and Enochs [9, 15] proved that every module admits a flat cover. For the past 5 years, my research has been deeply influenced by this result. The existence of flat covers in the category R-Mod is equivalent to the existence of cotorsion envelopes in R-Mod [65]. A left R-module C is *cotorsion* if $\mathrm{Ext}^1_R(F, C) = 0$ for every flat module ${}_RF$. In other words, every short exact sequence of the form

$$0 \longrightarrow C \longrightarrow Y \longrightarrow F \longrightarrow 0$$

is split exact, whenever F is a flat module. Cotorsion modules were first introduced by Harrison [30] in the setting of abelian groups. An abelian group C is cotorsion if and only if there exist abelian groups A and B such that

$$C = \mathrm{Ext}^1(A, B).$$

Every pure injective module is cotorsion. Indeed, we view the notion of a cotorsion module as a homologically defined generalization of a pure injective module. I applied the existence of cotorsion envelopes in a certain functor category [16] to prove the following.

Theorem 1. [35] *If $\mathcal{A}$ is an additive accessible category, then every object $M \in \mathcal{A}$ admits a pure injective envelope $m : M \to PE(M)$, which is a pure monomorphism.*

An additive accessible category [1, 11, 51] is the most general additive setting in which a theory of purity may be developed. The prototypical example of such a category is the category R-Flat of flat R-modules, and the pure injective objects of R-Flat are the flat cotorsion left R-modules. In a series of 5 articles [24, 25, 26, 27, 28], P.A. Guil Asensio and I have extended a large portion of the classical theory of pure injective modules to that of the pure injective objects of R-Flat. The main highlights of our theory include descriptions of:

The endomorphism ring: [24] If ${}_RC$ is a flat cotorsion R-module, then the endomorphism ring $S = \mathrm{End}_RC$ is von Neumann regular modulo the Jacobson radical $J = J(S)$. Furthermore, S/J is left self-injective and idempotents lift modulo J. In particular, if C is an indecomposable flat cotorsion module, then End_RC is local. (cf. [68])

Indecomposable pure injective objects: [27] Every ring has an indecomposable flat cotorsion R-module ${}_RC$. More generally, every flat left R-module admits a pure monomorphism into a product of flat cotorsion indecomposable modules.

The flat cover of a simple module: [27] If R is a semilocal ring with Jacobson radical J, then there is a commutative diagram

$$\begin{array}{ccc} & & R \\ & {}^{e}\swarrow & \downarrow{}^{q} \\ \mathrm{CE}(R) & \xrightarrow{\;c\;} & R/J, \end{array}$$

of left R-modules, where $q : R \to R/J$ is the natural quotient map, $e : R \to \mathrm{CE}({}_RR)$ is the cotorsion envelope, and $c : \mathrm{CE}(R) \to R/J$ is the flat cover. If $R/J = \oplus_{i=1}^{n} S_i$ is a direct sum of n simple modules, then $\mathrm{CE}(R) = \oplus_{i=1}^{n} C_i$ is a direct sum of n indecomposable flat cotorsion modules C_i with the property that $C_i/C_iJ \cong S_i$, and the natural quotient map $c_i : C_i \to C_i/C_iJ$ is the flat cover of S_i.

Σ-cotorsion modules: [26] If ${}_RC$ is a flat R-module with the property that the direct sum $C^{(I)}$ of any number of copies of C is cotorsion, then for every natural number n, the module ${}_RC$ satisfies the descending chain condition on n-ary *finite matrix subgroups* that are defined by *positive-primitive formulae* expressing divisibility conditions. (cf. [50, 20, 21, 67])

Triangulated categories and phantom morphisms. The notion of a phantom map, which has its origins in homotopy theory [52], was first introduced into the setting of a *triangulated category* [54] by Neeman [53]. Algebraic topologists [10] as well as those working in modular representation theory [7, 8, 22] noticed the relationship between purity and phantom morphisms. If R is a *Quasi-Frobenius* ring, as in the case of a finite group ring $k[G]$, then the *stable* category R-$\underline{\mathrm{mod}}$ of finitely presented modules modulo the morphisms that factor through a projective, is a triangulated category [12]. In my most recent article [37], I define a morphism $f : M \to N$ of R-modules to be *phantom* (where R is any associative ring) if for every morphism $g : A \to M$ with A finitely presented, the composition $fg : A \to M \to N$ factors through a projective module; and I prove the following.

Theorem 2. [37, Thm 7] *Given a left R-module N, there exists a short exact sequence*

$$0 \longrightarrow K \xrightarrow{\;k\;} Ph(\mathrm{N}) \xrightarrow{\;c\;} N \longrightarrow 0,$$

where the morphism $c : Ph(\mathrm{N}) \to \mathrm{N}$ is the phantom cover of N in R-Mod and K is a pure injective R-module.

One characterization of a flat module F asserts that every morphism $g : A \to F$ from a finitely presented module A factors through a projective. Phantom morphisms are therefore morphisms that behave like flat objects, and the theorem is an analogue of the existence of flat covers. The difference is measured by the kernel. The kernel of a flat cover is always cotorsion, while the kernel of a phantom cover belongs to the smaller class of pure injective modules. I have shown [37, Thm 10] that if the flat cover of an R-module ${}_RN$ has a pure injective kernel, then it is the phantom cover of N. In a joint project with P.A. Guil Asensio and B. Torecillas, I am involved with the further investigation of the relationship between pure injective modules and phantom covers.

The results on phantom covers grew out of the theory [36] that I developed for the functor category $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ of *contravariant* additive functors $\mathcal{F} : (R\text{-mod})^{\mathrm{op}} \to \mathrm{Ab}$. Historically, this functor category had been neglected in favor of its covariant sibling (mod-R, Ab). The covariant functor category (mod-R, Ab) has the desirable features that it is a locally coherent Grothendieck category [32]; that the module category R-Mod admits a fully faithful embedding ${}_R M \mapsto - \otimes_R M$; and that the injective objects have been characterized [23]. The main virtue of the contravariant functor category is the full and faithful embedding of R-Mod, via the functor $M \mapsto (-, M) := \mathrm{Hom}_R(-, M)$, onto the subcategory of flat objects of $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$. The main highlights of my theory include:

(1) a characterization of the flat cotorsion objects of $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ as those of the form $(-, M)$ where M is pure injective;

(2) the description of a minimal flat resolution of an object $\mathcal{F} \in ((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ as having the form

$$0 \longrightarrow (-, M) \xrightarrow{(-, f)} (-, N) \xrightarrow{(-, g)} (-, K) \xrightarrow{\pi} \mathcal{F} \longrightarrow 0,$$

where M and N are pure injective modules; and

(3) a characterization of the injective objects of the category $((R\text{-}\underline{\mathrm{mod}})^{\mathrm{op}}, \mathrm{Ab})$ as the functors of the form $\mathrm{Ext}^1(-, M)$ where M is a pure injective module.

The K-theory of the free abelian category. Given a ring R, consider the preadditive category $\mathbf{R} = \{*\}$ with one only object, whose endomorphism ring is R. An additive functor $M : \mathbf{R} \to \mathrm{Ab}$ is nothing more than a left R-module, whose underlying group structure is given by $M(*)$, with the action of R given by the $M(r)$. The *free abelian category* over R is an additive functor $\mathrm{Ab} : \mathbf{R} \to \mathrm{Ab(R)}$ into an abelian category Ab(R) satisfying the universal condition that every additive functor $\mathcal{M} : \mathbf{R} \to \mathcal{A}$ to an abelian category $\mathcal{A}$, extends to an exact functor as indicated,

$$\begin{array}{ccc} \mathrm{Ab(R)} & & \\ \uparrow & \searrow^{\mathrm{Ab}(\mathcal{M})} & \\ \mathbf{R} & \xrightarrow{\mathcal{M}} & \mathcal{A}, \end{array}$$

and furthermore, the extension $\mathrm{Ab}(\mathcal{M})$ is unique up to natural equivalence. The free abelian category may be thought of as an abelian category with a distinguished object $*$. It was introduced by Freyd [17], who proved its existence under much more general circumstances. Various descriptions of Ab(R) have been given by researchers from diverse fields: category theory [2], representation theory [3], the model theory of modules [31, 57], and the theory of derived categories [63].

The setting of the free abelian category over R permits a categorical generalization of the notion of the *annihilator ideal* $\mathrm{ann}_R(M)$ of a module M. This is carried out as follows. Since an R-module ${}_R M$ may be thought of as a functor $M : \mathbf{R} \to \mathrm{Ab}$, it extends uniquely to an exact functor $\mathrm{Ab(M)} : \mathrm{Ab(R)} \to \mathrm{Ab}$. The *Serre annihilator* of M is the subcategory $\mathcal{S}(M) \subseteq \mathrm{Ab(R)}$ that consists of those objects $G \in \mathrm{Ab(R)}$ for which $\mathrm{Ab(M)(G)} = 0$. The Serre annihilator $\mathcal{S}(M)$ is a *Serre subcategory* of Ab(R) in the sense that for any short exact sequence

$$0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 0$$

in Ab(R), $G_2 \in \mathcal{S}(M)$ if and only if G_1 and $G_3 \in \mathcal{S}(M)$. The Serre annihilator of M contains much more information than the annihilator ideal of M. For example, if k is an infinite field and R is a k-algebra, then two R-modules M and N are elementarily equivalent if and only $\mathcal{S}(M) = \mathcal{S}(N)$. In [33], I used the Serre annihilator of a certain module over an artin algebra Λ to prove that the

Krull-Gabriel dimension of Ab(Λ) cannot equal 1. It is also indispensible in my characterization [40], joint with Puninskaya, of strongly minimal modules over a commutative ring.

In his landmark paper [66] on the model theory of modules, Ziegler introduced a topology on the space of pure injective indecompsable left R-modules. This topological space Zg(R) is called the *Ziegler spectrum* of R. I discovered the following relationship between the Ziegler spectrum and the free abelian category.

Theorem 3. [32, Thm 3.8] *There is a bijective inclusion-preserving correspondence between the Serre subcategories $\mathcal{S}$ of the free abelian category Ab*(R) *and the open subsets $\mathcal{O}$ of the Ziegler spectrum Zg*(R)*. The correspondence is given by the rule*

$$\mathcal{S} \mapsto \mathcal{O}(\mathcal{S}) := \{U \in Zg(R) \mid \mathcal{S} \not\subseteq \mathcal{S}(U)\}.$$

Theorem 3 encapsulates in the category-theoretic idiom the most often applied form [66, Lemma 4.7] of Gödel's Compactness Theorem in the model theory of modules. If we regard the Cantor-Bendixson stratification of a topological space as an ascending filtration of open subsets, then under suitable conditions, the corresponding filtration of Serre subcategories of Ab(R) is given by the Krull-Gabriel filtration. There is also the notion of a Ziegler spectrum for a compactly generated triangulated category [5, 6, 48], but examples are lacking, and I am involved in a project with M. Prest on describing the Ziegler spectrum of the category of *homotopy spectra.* The results mentioned above, on the existence of flat cotorsion indecomposables, constitute an attempt to impose some sort of topological structure on the pure injective indecomposable objects of an additive accessible category.

In work that is still in preparation [38], I introduce a notion of homology for an abelian category $\mathcal{A}$ with a distinguished object. The free abelian category Ab(R) together with $*$ falls under that rubric. The main highlights of this theory of homology include:

(1) a natural isomorphism $H_0(\mathrm{Ab(R)}, *) \cong \mathrm{K}_0(\mathrm{Ab(R)})$ between 0-dimensional homology of the free abelian category and the Grothendieck group of Ab(R);
(2) for every Serre subcategory $\mathcal{S} \subseteq \mathrm{Ab(R)}$, a long exact sequence of homology that coincides in dimension 0 with Swan's long exact sequence [62] of K-groups;
(3) a description of H_1 in case $*$ is of finite length.

As in the theory of *motives* [29], the definition of the homology relies on the fact that the objects of Ab(R) may be expressed by formulae in a first-order language.

Pseudo-finite dimensional representations of Lie algebras, etc. Let k be an algebraically closed field of characteristic 0 and denote by L the *Lie algebra* $\mathrm{sl}(2, k)$ of 2×2 matrices over k having trace 0. The *universal enveloping algebra* $U(L)$ of L is an associative ring whose category of modules $U(L)$-Mod is equivalent to the category of representations of L. The standard action of L on the *affine plane* $k[x, y]$ extends to an action of $U(L)$. Under that action, the affine plane admits a direct sum decomposition

$$k[x, y] = \oplus_{d \geq 0}\, k[x, y]_d,$$

where $k[x, y]_d$ denotes the submodule of homogeneous polynomials $p(x, y)$ of degree d. The article [34] is devoted to a detailed analysis of the situation when the free abelian category Ab(U(L)) is localized at the Serre annihilator of the $U(L)$-module $k[x, y]$.

Theorem 4. *The localization Ab*(U(L))/$\mathcal{S}$(k[x, y]) *has projective global dimension* 0.

The heuristic significance of Theorem 4 may be explained as follows. Weyl's Theorem [42] for a simple Lie algebra over k implies that every finite dimensional representation of $U(L)$ is semisimple. Furthermore, it is known that every simple finite dimensional representation of $U(L)$ is isomorphic to $k[x, y]_d$ for some $d \geq 0$. If there were only finitely many simple finite dimensional representations, one could factor out by their annihilator ideal $I \subseteq U(L)$ and obtain a semisimple artinian ring $U(L)/I$, whose category of finite length modules would be equivalent to the category of finite dimensional representations of L. Alas, there are *infinitely* many simple finite dimensional representations, and a general theorem of Harish-Chandra [14] implies that the annihilator ideal of $k[x, y]$ is 0. Theorem 4 rectifies the situation, for it implies the existence of a *von Neumann regular ring* $U'(L)$ and an epimorphism of rings $\rho : U(L) \to U'(L)$ that induces an equivalence

$$\mathrm{Ab(U(L))}/\mathcal{S}(\mathrm{k[x,y]}) \cong \mathrm{Ab(U'(L))}$$

of categories. Instead of factoring out by the annihilator ideal, we localize at the Serre annihilator; instead of a factor ring of $U(L)$, we obtain a ring epimorphism $\rho : U(L) \to U'(L)$; and instead of a semisimple ring, whose modules are all projective, we get a von Neumann regular ring $U'(L)$, whose modules are all flat.

The epimorphism $\rho : U(L) \to U'(L)$ induces a homeomorphism [58] from the Ziegler spectrum of $U'(L)$ and the closure of the finite dimensional simple representations $k[x, y]_d$ in the Ziegler spectrum of $U(L)$. It also implies that the full subcategory of $U(L)$-modules obtained by restriction of scalars along ρ is a category, equivalent to $U'(L)$-Mod, whose objects form a class axiomatizable by sentences from the language of left R-modules. Every finite dimensional representation of $U(L)$ admits the structure of a $U'(L)$-module, and therefore satisfies these axioms. One more additional axiom schema [34, Cor 64] suffices to provide an axiomatization of the finite dimensional representations of $U(L)$. A representation N is called *pseudo-finite dimensional* if it satisfies the axioms for a finite dimensional representation. The only simple representation of the von Neumann regular ring $U'(L)$ which is not pseudo-finite dimensional is the field of fractions of $U(L)$.

The classification of finite dimensional representations of $\mathrm{sl}(2, k)$ is essential to the classification of finite dimensional representations of any simple Lie algebra over k, as well as of their Lie groups and quantum deformations [43, 46]. The broader goal of my work in this area is to extend the results above in these directions as well as for the category of *integrable* representations over a *Kac-Moody* algebra [44, 41]. In a joint project still in preparation, S. L'Innocente and I [39] are developing an analogous theory for the quantum plane $k_q[x, y]$, when q is not a root of unity; as well as the much more difficult question of the affine plane $k[x, y]$ considered as a representation of the Lie group $\mathrm{GL}(2, k)$.

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